

# $\theta$ -CURVE POLYNOMIALS AND FINITE-TYPE INVARIANTS

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**ABSTRACT.** The normalized Yamada polynomial,  $\tilde{R}_A$ , is a polynomial invariant in variable  $A$  for  $\theta$ -curves. In this work, we show that the coefficients of  $\tilde{R}_{e^x}$  which is obtained by replacing  $A$  with  $e^x = \sum x^n/n!$  are finite-type invariants for  $\theta$ -curves although the coefficients of original  $\tilde{R}_A$  are not finite-type. A similar result can be obtained in the case of Yokota polynomial for  $\theta$ -curves.

## 1. INTRODUCTION

Birman and Lin discovered infinitely many finite type invariants for knots derived from polynomial<sup>1</sup> invariants [BL]. They showed that the coefficients of one variable HOMFLY polynomial and Kauffman polynomial are finite-type with substituting the variable  $t$  by  $e^x$ . Bar-Natan showed that the coefficients of Conway polynomial are also finite-type [B]. On the other hand, Zhu observed that the coefficients of Jones polynomial are not finite-type [Zhu]. Using Zhu's idea, Jin and Lee showed that the coefficients of the 2-variable HOMFLY polynomial, the 2-variable Kauffman polynomial and the Q-polynomial are not finite type invariants [JL].

In this paper, following the studies listed above, we examine whether the coefficients of two polynomial invariants of  $\theta$ -curves are finite-type invariants. A  $\theta$ -curve is a graph embedded in  $\mathbf{R}^3$  consisting of two vertices and three edges between them.

Originally, finite-type invariants were defined for knots. Later, Stanford extended them to links and some other spatial graphs including  $\theta$ -curves and Kanenobu investigated them with emphasis on  $\theta$ -curves [S1, S2, Kan]. In fact, abundant finite-type invariants for  $\theta$ -curves can be obtained in the following way: There is a well-defined 3-component link associated with each  $\theta$ -curve, which is the boundary of a surface obtained by thickening the  $\theta$ -curve in a canonical way [KSWZ]. Every finite-type invariant for this associated link was shown to be a finite type invariant for the original  $\theta$ -curve [S2]. Therefore all the finite type invariants derived from the polynomial invariants of the 3-component links associated with  $\theta$ -curves are finite type invariants.

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2000 *Mathematics Subject Classification.* 57M15, 05C10.

*Key words and phrases.* spatial graph,  $\theta$ -curve, Yamada polynomial, Yokota polynomial, finite-type invariant.

<sup>1</sup>In this article, a polynomial stands for a Laurent polynomial.

On the other hand, there are some invariants for  $\theta$ -curves which seem to have different origins from the above. One of them is Yamada polynomial [Ya]. There is no known relation between the Yamada polynomial of a  $\theta$ -curve and invariants of its associated 3-component link. Yamada polynomial can be calculated by skein relations from diagrams while this method doesn't seem to work for the invariants of  $\theta$ -curves coming from their associated 3-component links.

Let  $\tilde{R}_A$  be the Yamada polynomial for  $\theta$ -curves normalized to behave multiplicatively under connected sums. We will show that the coefficients of  $\tilde{R}_{e^x}$  which is obtained by replacing the variable  $A$  with  $e^x = \sum_{n=0}^{\infty} x^n/n!$  are finite-type invariants for  $\theta$ -curves although the coefficients of the original  $\tilde{R}_A$  are not finite-type. In 1999, Yokota introduced a polynomial invariant for  $\theta_m$ -curves [Yo]. A  $\theta_m$ -curve is a spatial graph consisting of 2 vertices and  $m$ -edges between them. This polynomial is obtained from the  $SU(m)$  invariant of a linear combination of links constructed from a given  $\theta$ -curve diagram. This method is different from that of [KSWZ]. For the coefficients of Yokota polynomial, we obtain the same results as in the case of Yamada polynomial.

## 2. DEFINITIONS

A *spatial graph* is a finite graph which is PL embedded in  $\mathbf{R}^3$ . Two spatial graphs  $G$  and  $G'$  are said to be *equivalent* or *isotopic* if there exists an orientation preserving autohomeomorphism of  $\mathbf{R}^3$  carrying one onto the other. A *projection* of a spatial graph is its image under a natural projection map of  $\mathbf{R}^3$  onto a euclidean plane. We always assume that spatial graphs are in general position with the projection maps so that the only singular points of the projections are transverse double points away from vertices. A double point in a projection is called a *crossing*. A *diagram* of a spatial graph is its projection with informations denoting which strand is over or under at each crossing. See Figure 1. Two spatial graphs are equivalent if and only if a diagram of one of them can be transformed to a diagram of the other by a finite sequence of moves I–VI given in Figure 2 [Kau].

A  $\theta$ -*curve* is a spatial graph consisting of two vertices and 3 edges joining them. A  $\theta$ -curve is said to be *trivial* if it is equivalent to a  $\theta$ -curve in  $\mathbf{R}^2$ . A crossing in a diagram of a  $\theta$ -curve is called a *positive crossing* (resp. *negative crossing*) or said to have *crossing number* '+1' (resp. '-1') if it is like the one in Figure 3 with '+' sign (resp. '-' sign) when all the edges are oriented coherently from one vertex to the other.

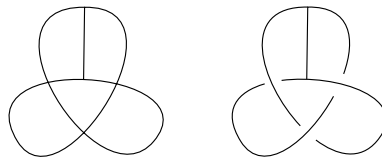


FIGURE 1. Projection and diagram of  $3_1$

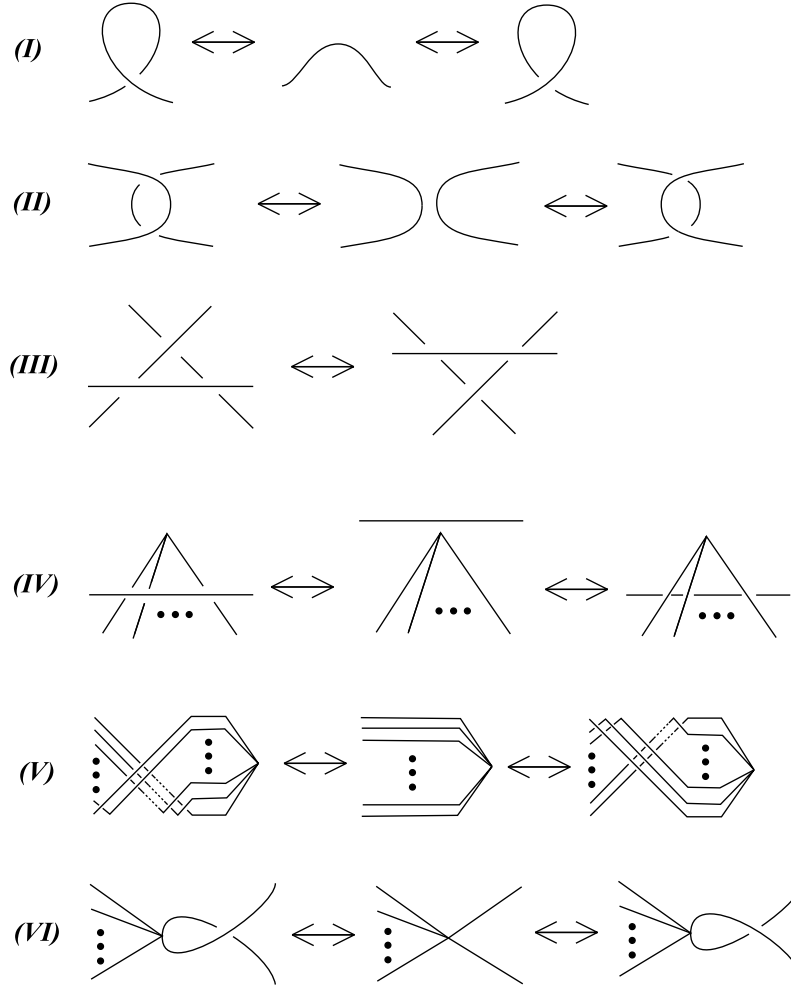
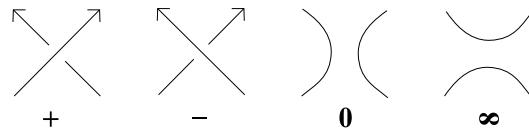


FIGURE 2. Moves


FIGURE 3. Positive, negative crossings and  $0, \infty$  changes

For a positive integer  $n$ , an  $n$ -sign is an  $n$ -tuple of  $1, -1, 0$  and  $\infty$ , i.e., an element  $(\epsilon_1, \dots, \epsilon_n) \in \{1, -1, 0, \infty\}^n$ . Let  $C = \{c_1, c_2, \dots, c_n\}$  be a set of  $n$  distinct crossings in a diagram  $D$  of a  $\theta$ -curve and let  $\epsilon = (\epsilon_1, \dots, \epsilon_n)$  be an  $n$ -sign. Denote by  $D_\epsilon$  the diagram obtained from  $D$  by replacing each crossing  $c_i$  with a local diagram of Figure 3 corresponding to  $\epsilon_i$ .

For any rational invariant  $v$  of  $\theta$ -curves, we define

$$v(D \mid C) = \sum_{\epsilon \in \{1, -1\}^n} (-1)^{|\epsilon|} v(D_\epsilon)$$

where  $|\epsilon| = \sum_{\epsilon_i > 0} \epsilon_i$ . We say that  $v$  is *finite-type of order less than  $n$*  if  $v(D \mid C) = 0$  for every set  $C$  of  $n$  distinct crossings in every  $\theta$ -curve diagram  $D$ . If this vanishing condition is true for any  $n + 1$  crossings but not for some  $n$  crossings, we say that  $v$  is *finite-type of order  $n$* .

Given two  $\theta$ -curves  $\Theta$  and  $\Theta'$ , we may assume that  $\Theta$  is in the upper half space  $\mathbf{R}_+^3 = \{(x, y, z) \in \mathbf{R}^3 \mid z \geq 0\}$  except for a small neighborhood of one vertex,  $\Theta'$  is in the lower half space  $\mathbf{R}_-^3 = \{(x, y, z) \in \mathbf{R}^3 \mid z \leq 0\}$  except for a small neighborhood of one vertex,  $\Theta \cap \mathbf{R}_0^2 = \Theta' \cap \mathbf{R}_0^2 = \{3 \text{ points}\}$  where  $\mathbf{R}_0^2 = \{(x, y, z) \in \mathbf{R}^3 \mid z = 0\}$ , and  $(\Theta \cap \mathbf{R}_-^3) \cup (\Theta' \cap \mathbf{R}_+^3)$  is a trivial  $\theta$ -curve. Then  $(\Theta \cap \mathbf{R}_+^3) \cup (\Theta' \cap \mathbf{R}_-^3)$  is a  $\theta$ -curve which is called a *connected sum* of  $\Theta$  and  $\Theta'$ , denoted by  $\Theta \# \Theta'$ .

### 3. YAMADA POLYNOMIAL

In [Ya], Yamada introduced a polynomial invariant in variable  $A$  for diagrams of spatial graphs, which is invariant under the moves II–IV in Figure 2 and is a flat isotopy<sup>2</sup> invariant up to a multiplication by a power of  $-A$ . Two diagrams of spatial graphs are said to be *flat isotopic* if one can be transformed into the other by a finite sequence of moves I–V in Figure 2. For a diagram  $D$  of a  $\theta$ -curve, let  $R_A(D)$  be the polynomial invariant of Yamada for  $D$ . Then  $R_A$  can be determined by the moves II–IV and the formulae Ya1–Ya5, where  $\ominus$ ,  $\bigcirc\text{---}\bigcirc$  and ' $D \bigcirc$ ' denote a diagram of a  $\theta$ -curve in a plane, that of a handcuff curve<sup>3</sup> in a plane and the diagram  $D$  with a disjoint unknotted circle, respectively, and  $\sigma = A + 1 + A^{-1}$ .

$$(Ya1) \quad R_A(\ominus) = \sigma - \sigma^2 = -(A^2 + A + 2 + A^{-1} + A^{-2})$$

$$(Ya2) \quad R_A\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) - R_A\left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}\right) = (A - A^{-1}) \left( R_A\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) - R_A\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) \right)$$

$$(Ya3) \quad R_A\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right) = A^2 R_A\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right), \quad R_A\left(\begin{array}{c} \diagdown \\ \diagup \end{array}\right) = A^{-2} R_A\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$$

$$(Ya4) \quad R_A\left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}\right) = -A^{-3} R_A\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right), \quad R_A\left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}\right) = -A^3 R_A\left(\begin{array}{c} \diagup \\ \diagdown \end{array}\right)$$

$$(Ya5) \quad R_A(\bigcirc\text{---}\bigcirc) = 0$$

$$(Ya6) \quad R_A(D \bigcirc) = \sigma R_A(D)$$

A crossing in a diagram is called a *self-crossing* if its two strands are from a single edge, and a *non-self-crossing* otherwise. For a diagram  $D$  of a  $\theta$ -curve, let  $s(D)$  and  $n(D)$  denote the sums of the crossing numbers of the self-crossings in  $D$  and the non-self-crossings in  $D$ , respectively.

**Proposition 3.1.** *For any  $\theta$ -curve diagram  $D$ , we define*

$$\tilde{R}_A(D) = (-A)^{n(D)-2s(D)} R_A(D) / (\sigma - \sigma^2).$$

*Then  $\tilde{R}_A$  is an isotopy invariant of  $\theta$ -curves with the following properties:*

1.  $\tilde{R}_A(D)$  is a polynomial in variable  $A$ .
2.  $\tilde{R}_A(\Theta \# \Theta') = \tilde{R}_A(\Theta) \tilde{R}_A(\Theta')$  for two  $\theta$ -curves  $\Theta, \Theta'$ .
3. If  $\bar{D}$  is the mirror image of  $D$ , then  $\tilde{R}_A(\bar{D}) = \tilde{R}_{A^{-1}}(D)$ .

<sup>2</sup>For  $\theta$ -curves, flat isotopy implies isotopy, since the the move VI at a trivalent vertex can be derived from the moves I–V.

<sup>3</sup>A handcuff curve is a connected spatial graph with two vertices and three edges obtained by connecting two disjoint loop edges with the third edge.

*Proof.* According to [Ya, Theorem 7], the polynomial  $(-A)^{n-2s}R_A$  is an isotopy invariant of  $\theta$ -curves. Therefore so is  $\tilde{R}_A$ . The property ‘1’ is a consequence of the fact that  $R_A$  is a multiple of  $\sigma - \sigma^2$  for any  $\theta$ -curve diagram  $D$ , which is not hard to see. The property ‘2’ follows from the connected sum formula

$$R_A(D \sharp D') = R_A(D)R_A(D')/(\sigma - \sigma^2)$$

for any  $\theta$ -curve diagrams  $D$  and  $D'$  [Ya, Theorem 5]. Finally, the property ‘3’ is a consequence of the fact  $R_A(\bar{D}) = R_{A^{-1}}(D)$  [Ya, Proposition 6].  $\square$

**Theorem 3.2.** *For any integer  $n$ , the coefficient of  $x^n$  in the power series  $\tilde{R}_{e^x}$ , obtained from  $\tilde{R}_A$  by the substitution  $A = e^x = \sum_{i=0}^{\infty} x^i/i!$ , is a finite-type invariant of order at most  $n$ . On the other hand, the coefficients of the polynomial  $\tilde{R}_A$ , in variable  $A$ , are not finite-type invariants.*

*Proof.* Let  $D$  be a  $\theta$ -curve diagram. Suppose  $D_+$ ,  $D_-$ ,  $D_0$  and  $D_\infty$  are the diagrams with one crossing of  $D$ , say  $c$ , replaced by the local diagrams of Figure 3, respectively. Then we have

$$\begin{aligned}\tilde{R}_A(D_+) &= (-A)^m R_A(D_+)/(\sigma - \sigma^2) \\ \tilde{R}_A(D_-) &= (-A)^{m+j} R_A(D_-)/(\sigma - \sigma^2),\end{aligned}$$

where  $m = m(D_+) = n(D_+) - 2s(D_+)$  and

$$j = j(c) = \begin{cases} 4 & \text{if } c \text{ is a self-crossing,} \\ -2 & \text{if } c \text{ is a non-self-crossing.} \end{cases}$$

In the following computations, we omit the subscript  $A$  from  $\tilde{R}_A$  and  $R_A$ .

Using (Ya2), we obtain

$$\begin{aligned}(\sigma - \sigma^2)(\tilde{R}(D_+) - \tilde{R}(D_-)) \\ (E1) \quad &= (-A)^m (R(D_+) - A^j R(D_-)) \\ &= (-A)^m ((1 - A^j)R(D_-) + (A - A^{-1})(R(D_0) - R(D_\infty))).\end{aligned}$$

Suppose  $D_+$ ,  $D_-$ ,  $D_0$  and  $D_\infty$  are the diagrams obtained from  $D$  by a repeated application of the skein relation (Ya2), which are identical except at one place where they differ as indicated in Figure 3. The local orientations of  $D_+$  and  $D_-$  shown in Figure 3 are induced from the orientation of  $D$  in which all edges are oriented from one vertex to the other. Using (Ya2) again, we have

$$\begin{aligned}(E2) \quad &R(D_+) - A^t R(D_-) \\ &= R(D_-) + (A - A^{-1})(R(D_0) - R(D_\infty)) - A^t R(D_-) \\ &= (1 - A^t)R(D_-) + (A - A^{-1})(R(D_0) - R(D_\infty)),\end{aligned}$$

for any nonzero integer  $t$ .

For a set  $C = \{c_1, c_2, \dots, c_n\}$  of  $n$  distinct crossings of  $D$ , we consider the sum

$$(E3) \quad (\sigma - \sigma^2)\tilde{R}(D \mid C) = \sum_{\epsilon \in \{1, -1\}^n} (-1)^{|\epsilon|} (\sigma - \sigma^2)\tilde{R}(D_\epsilon).$$

For  $1 \leq k \leq n$ , let  $E_k = \{1, -1\}^k$  and  $F_k = \{-1, 0, \infty\}^k$ . For a  $k$ -sign  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{1, -1, 0, \infty\}^k$ , let  $\epsilon+$ ,  $\epsilon-$ ,  $\epsilon 0$  and  $\epsilon \infty$  be the  $(k+1)$ -signs obtained from  $\epsilon$  by appending 1,  $-1$ , 0 and  $\infty$ , respectively. Let  $f_k = -(1 - A^{j(c_k)})$  and  $f_0 = -f_\infty = -(A - A^{-1})$ . For each  $k$ -sign  $\delta = (\delta_1, \dots, \delta_k) \in F_k$ , with  $1 \leq k \leq n$ , we define

$$G_i(\delta) = \begin{cases} f_{n+1-i} & \text{if } \delta_i = -1 \\ f_0 & \text{if } \delta_i = 0 \\ f_\infty & \text{if } \delta_i = \infty, \end{cases}$$

for  $1 \leq i \leq k$ . Applying (E1) to the right hand side of (E3), we obtain

$$\begin{aligned} & \sum_{\epsilon \in E_n} (-1)^{|\epsilon|} (\sigma - \sigma^2) \tilde{R}(D_\epsilon) \\ &= \sum_{\epsilon \in E_{n-1}} (-1)^{|\epsilon+|} (\sigma - \sigma^2) (\tilde{R}(D_{\epsilon+}) - \tilde{R}(D_{\epsilon-})) \\ (E4) \quad &= \sum_{\epsilon \in E_{n-1}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+})} (f_n R(D_{\epsilon-}) + f_0 R(D_{\epsilon 0}) + f_\infty R(D_{\epsilon \infty})) \\ &= \sum_{\delta \in F_1} G_1(\delta) \sum_{\epsilon \in E_{n-1}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+})} R(D_{\epsilon \delta}). \end{aligned}$$

If  $\epsilon \in E_{n-k-1}$ , then

$$\begin{aligned} |\epsilon+| &= |\epsilon-| + 1 = |\epsilon| + 1, \\ m(D_{\epsilon-+\dots+}) &= m(D_{\epsilon++\dots+}) + j(c_{n-k}), \end{aligned}$$

where ‘ $+\dots+$ ’ means a duplication of  $+$ ’s so that the length of the whole ‘sign’ is  $n$ . Using (E2) with a  $k$ -sign  $\delta \in F_k$ , we obtain

$$\begin{aligned} & \sum_{\epsilon \in E_{n-k}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+\dots+})} R(D_{\epsilon \delta}) \\ &= \sum_{\epsilon \in E_{n-k-1}} (-1)^{|\epsilon+|} (-A)^{m(D_{\epsilon+\dots+})} (R(D_{\epsilon+\delta}) - A^{j(c_{n-k})} R(D_{\epsilon-\delta})) \\ &= \sum_{\epsilon \in E_{n-k-1}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+\dots+})} (f_{n-k} R(D_{\epsilon-\delta}) + f_0 R(D_{\epsilon 0 \delta}) + f_\infty R(D_{\epsilon \infty \delta})) \\ &= \sum_{\delta' \in F_1} G_{k+1}(\delta') \sum_{\epsilon \in E_{n-k-1}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+\dots+})} R(D_{\epsilon \delta' \delta}). \end{aligned}$$

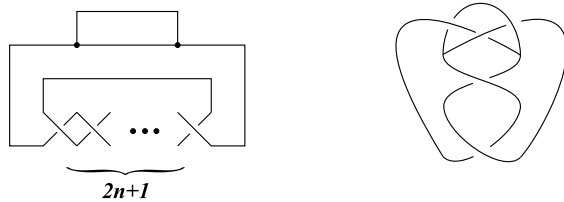


FIGURE 4.  $T_{2n+1}$  and  $5_1$

Repeated applications of this to (E4) lead us to

$$\begin{aligned}
 & \sum_{\epsilon \in E_n} (-1)^{|\epsilon|} (\sigma - \sigma^2) \tilde{R}(D_\epsilon) \\
 &= \sum_{\delta \in F_2} G_1(\delta) G_2(\delta) \sum_{\epsilon \in E_{n-2}} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon++})} R(D_{\epsilon\delta}) \\
 &= \dots \\
 &= \sum_{\delta \in F_{n-1}} \prod_{i=1}^{n-1} G_i(\delta) \sum_{\epsilon \in E_1} (-1)^{|\epsilon|} (-A)^{m(D_{\epsilon+\dots+})} R(D_{\epsilon\delta}) \\
 &= (-A)^{m(D_{+\dots+})} \sum_{\delta \in F_n} \prod_{i=1}^n G_i(\delta) R(D_\delta).
 \end{aligned}$$

This shows that  $(\sigma - \sigma^2) \tilde{R}(D \mid C)$  is divisible by  $(1 - A)^n$ , since each  $G_i(\delta)$  is divisible by  $1 - A$ . Therefore, after the substitution  $A = e^x = \sum_{i=0}^{\infty} x^i / i!$ , the power series  $\tilde{R}_{e^x}(D \mid C)$  has no terms of degree less than  $n$ . This proves the first part of the theorem.

Let  $v_r(\Theta)$  be the coefficient of  $A^r$  in  $\tilde{R}(\Theta)$  for a  $\theta$ -curve  $\Theta$ . To show that  $v_r$  is not finite-type for any integer  $r$ , we use the  $\theta$ -curves  $T_{2n+1}$  with  $n \geq 0$  and  $5_1$  shown in Figure 4.

Since  $T_1$  is trivial, we have  $\tilde{R}(T_1) = 1$ . Inductively, we can compute the maximal and the minimal degrees of  $\tilde{R}(T_{2n+1})$ . For  $n \geq 1$ , they are  $-2n$  and  $-(8n + 3)$ , respectively. For  $n \geq 1$ , let  $C_{2n}$  be a set of  $2n$  distinct crossings of  $T_{4n+1}$ . The following computation shows that  $v_0$  is not finite-type.

$$v_0(T_{4n+1} \mid C_{2n}) = \sum_{p=0}^{2n} (-1)^p \binom{2n}{p} v_0(T_{4n-2p+1}) = 1 \neq 0.$$

Let  $G = \bar{T}_5 \# 5_1$ . Since the minimal degree<sup>4</sup> of  $\tilde{R}(5_1)$  is  $-3$ , that of  $\tilde{R}(G) = \tilde{R}(\bar{T}_5) \tilde{R}(5_1) = \tilde{R}_{A^{-1}}(T_5) \tilde{R}(5_1)$  is  $1$ . For a positive integer  $r$ , let  $G^r$  be a connect sum of  $r$  copies of  $G$ . Since the minimal degree of

$$\tilde{R}(G^r \# \bar{T}_{2n+1}) = (\tilde{R}(G))^r \tilde{R}(\bar{T}_{2n+1})$$

is  $r + 2n$ , we see that  $v_r(G^r \# \bar{T}_{2n+1}) \neq 0$  if and only if  $n = 0$ . This leads us to

$$v_r(G^r \# \bar{T}_{4n+1} \mid C_{2n}) = \sum_{p=0}^{2n} (-1)^p \binom{2n}{p} v_r(G^r \# \bar{T}_{4n-2p+1}) \neq 0,$$

for any  $n \geq 1$ . Considering the mirror images and the maximal degrees, we also obtain

$$v_{-r}(\bar{G}^r \# T_{4n+1} \mid C_{2n}) = \sum_{p=0}^{2n} (-1)^p \binom{2n}{p} v_{-r}(\bar{G}^r \# T_{4n-2p+1}) \neq 0,$$

for any  $n \geq 1$ . This proves the second part of the theorem.  $\square$

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<sup>4</sup> $\tilde{R}(5_1) = A^{10} - 2A^9 - 2A^8 + 6A^7 - 2A^6 - 6A^5 + 8A^4 + A^3 - 7A^2 + 3A + 3 - 3A^{-1} + A^{-3}$ .

4. YOKOTA POLYNOMIAL FOR  $\theta_m$ -CURVES

In [Yo], Yokota introduced a polynomial invariant for  $\theta_m$ -curves. It is a normalization of the  $SU(m)$  invariant of a linear combination of link diagrams derived from a given  $\theta_m$ -curve diagram  $D$  in a way which is different from that of [KSWZ]. The polynomial has also some properties about local changes in diagram which enable us to compute it. Among them, we introduce only what is necessary for  $\theta$ -curves. Let  $\langle D \rangle$  be the  $SU(3)$  invariant derived from a  $\theta$ -curve diagram  $D$ . We assume that the edges of  $D$  are oriented coherently from one vertex to the other. Then  $\langle D \rangle$  is a polynomial in variable  $t$  which is invariant under the moves II–IV and is determined by the following formulae:

$$\begin{aligned}
(\text{Yo1}) \quad & \langle \ominus \rangle = 1 \\
(\text{Yo2}) \quad & t \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle - t^{-1} \left\langle \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} \right\rangle = (t^3 - t^{-3}) \left\langle \begin{array}{c} \nearrow \\ \nwarrow \end{array} \right\rangle \left\langle \begin{array}{c} \nwarrow \\ \nearrow \end{array} \right\rangle \\
(\text{Yo3}) \quad & t^{-8} \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle = \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle = t^8 \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle \\
(\text{Yo4}) \quad & -t^4 \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle = \left\langle \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} \right\rangle = -t^{-4} \left\langle \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} \right\rangle \\
(\text{Yo5}) \quad & \langle D \bigcirc \rangle = (t^6 + 1 + t^{-6}) \langle D \rangle
\end{aligned}$$

**Proposition 4.1.** [Yo] *For any  $\theta$ -curve diagram  $D$ , we define*

$$P(D) = (-t^4)^{n(D)-2s(D)} \langle D \rangle,$$

*Then,  $P_z$ , obtained from  $P$  by the substitution  $z = t^3$ , is an isotopy invariant for  $\theta$ -curves with the following properties:*

1.  $P_z(D)$  is a polynomial in variable  $z$ .
2.  $P_z(\Theta \# \Theta') = P_z(\Theta) P_z(\Theta')$  for two  $\theta$ -curves  $\Theta$  and  $\Theta'$ .
3. If  $\bar{D}$  is the mirror image of  $D$ , then  $P_z(\bar{D}) = P_{z^{-1}}(D)$ .

**Theorem 4.2.** *For any integer  $n$ , the coefficient of  $x^n$  in the power series  $P_{e^x}$ , obtained from  $P_z$  by the substitution  $z = e^x = \sum_{i=0}^{\infty} x^i / i!$ , is a finite-type invariant of order at most  $n$ . On the other hand, the non-trivial coefficients, i.e., those of even degree terms, of the polynomial  $P_z$  are not finite-type invariants.*

*Proof.* The first part can be proven in a similar way to that of Theorem 3.2, using the skein relation (Yo2). It is easily seen that  $P_z$  has only even degree terms. The maximal degree and the minimal degree of  $P_z(T_{2n+1})$  are  $-4n$  and  $-(8n+4)$ , respectively, for  $n \geq 1$ , and those<sup>5</sup> of  $P_z(3_1)$  are 10 and 2, respectively. The second part can be proven in a similar way to that of Theorem 3.2, using  $3_1^r \# \bar{T}_{4n+1}$  instead of  $G^r \# \bar{T}_{4n+1}$ .  $\square$

## ACKNOWLEDGEMENT

The first author would like to thank Yoshiyuki Yokota for helpful answers to his questions.

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<sup>5</sup> $P_z(3_1) = z^2 + z^8 - z^{10}$ .



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